

## Lines Imply Spaces in Density Ramsey Theory

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Some results of geometric Ramsey theory assert that if  $F$  is a finite field (respectively, set) and  $n$  is sufficiently large, then in any coloring of the points of  $F^n$  there is a monochromatic  $k$ -dimensional affine (respectively, combinatorial) subspace (see [9]). We prove that the density version of this result for lines (i.e.,  $k = 1$ ) implies the density version for arbitrary  $k$ . By using results in [3, 6] we obtain various consequences: a “group-theoretic” version of Roth’s Theorem, a proof of the density assertion for arbitrary  $k$  in the finite field case when  $|F| = 3$ , and a proof of the density assertion for arbitrary  $k$  in the combinatorial case when  $|F| = 2$ .

## 1. RESULTS

In this section we will state and discuss the main results and prove some corollaries. The proofs of the main results are in the following section. Throughout  $q$  denotes a prime power.

Let  $\mathbf{F}_q$  be the field with  $q$  elements and let  $V$  be an  $n$ -dimensional vector space over  $\mathbf{F}_q$ . For each positive integer  $k$  and positive real number  $\varepsilon$  let  $n(\varepsilon, k, q)$  denote the smallest integer (if one exists) such that

$$n = \dim_{\mathbf{F}_q} V \geq n(\varepsilon, k, q), \quad A \subset V, \quad |A| > \varepsilon |V|,$$

imply that  $A$  contains an affine  $k$ -space. (By an affine  $k$ -space we mean any translate of a  $k$ -dimensional vector subspace; the purist will note that we only use the structure of  $V$  as an affine space.)

The “Affine Line Conjecture” is the assertion that  $n(\varepsilon, 1, q)$  exists for all

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$\varepsilon > 0$  and all  $q$ . The existence of  $n(\varepsilon, k, q)$  would be a density version of the results in [9] on Ramsey theorems in geometric contexts.

The main assertion of this paper is that if, for a fixed  $q$ ,  $n(\varepsilon, 1, q)$  exists for all  $\varepsilon > 0$ , then  $n(\varepsilon, k, q)$  exists for all  $k$  and all  $\varepsilon > 0$ . We will also reinterpret this result in the context of "combinatorial"  $k$ -spaces and "lattices" in abelian groups. We include a number of corollaries and remarks.

(It is not hard to see that if  $n(\varepsilon, 1, q)$  exists for all  $\varepsilon > 0$  and all  $q$ , then  $n(\varepsilon, k, q)$  exists for all  $k$ ,  $\varepsilon$ , and  $q$ . Indeed, if  $\varepsilon$ ,  $k$ , and  $q$  are given, let  $F$  be the extension of  $\mathbf{F}_q$  of degree  $k$ . An affine line in an  $F$ -vector space is a  $k$ -space over  $\mathbf{F}_q$  if we "restrict scalars" to  $\mathbf{F}_q$ ; from this it is easy to see that the existence of an affine line in a large enough subset of  $F^n$  implies the existence of an affine  $k$ -space in a large enough subset of  $\mathbf{F}_q^{kn}$ .)

**THEOREM 1.** *Suppose that  $\mathbf{F}_q$  is a fixed finite field and that  $n(\varepsilon, 1, q)$  exists for all  $\varepsilon > 0$ . Then  $n(\varepsilon, k, q)$  exists for all  $\varepsilon > 0$  and all  $k$ .*

**COROLLARY.** *The integers  $n(\varepsilon, k, 2)$  and  $n(\varepsilon, k, 3)$  exist for all  $\varepsilon > 0$  and all  $k$ .*

*Proof of the corollary.* Any two-element subset of an  $\mathbf{F}_2$  vector space is an affine line so it is trivial that  $n(\varepsilon, 1, q)$  exists. The theorem then implies that  $n(\varepsilon, k, 2)$  exists for all  $k$  (see the corollary to Lemma 1 in [3] for a different proof of the existence of  $n(\varepsilon, k, 2)$ ). The existence of  $n(\varepsilon, k, 3)$  follows from Theorem 1 and the existence of  $n(\varepsilon, 1, 3)$  which is the central result of [3]. This finishes the proof of the corollary.

A set  $\{x_1, \dots, x_k\}$  of the elements in an abelian group  $G$  is said to be independent if  $c_1x_1 + c_2x_2 + \dots + c_kx_k = 0$  implies that  $c_ix_i = 0$  for each  $i$ . An  $(m, k)$ -lattice in an abelian group  $G$  is a set of the form

$$M = \{a + c_1x_1 + \dots + c_kx_k : c_i = 0, 1, \dots, m-1\},$$

where  $a$  is an element of  $G$  and the  $x_i$  are independent. If  $V$  is a vector space over a finite field, then by an  $(m, k)$ -lattice in  $V$  we mean an  $(m, k)$ -lattice in its underlying additive group.

Let  $n'(\varepsilon, k, q)$  denote the smallest integer (if one exists) such that if

$$n = \dim_{\mathbf{F}_q} V \geq n'(\varepsilon, k, q), \quad A \subset V, \quad |A| > \varepsilon |V|,$$

then  $A$  contains a  $(3, k)$ -lattice.

**THEOREM 2.**  *$n'(\varepsilon, k, q)$  exists for all  $\varepsilon > 0$ ,  $k$ , and  $q$ .*

**COROLLARY.** *For each  $\varepsilon > 0$  and positive integer  $k$  there is an integer  $m(\varepsilon, k)$  such that if  $G$  is any finite abelian group with more than  $m(\varepsilon, k)$  elements and  $A$  is any subset of  $G$  with more than  $\varepsilon |G|$  elements, then there is a  $(3, k)$ -lattice inside  $A$ .*

*Proof of the corollary.* Let  $k$  and  $\varepsilon$  be given. Choose by Szemerédi's theorem [10] a large enough  $n$  so that any subset of  $\{1, 2, \dots, n\}$  with more than  $\varepsilon n$  elements contains an arithmetic progression with  $3^k$  terms. Choose  $m(\varepsilon, k)$  large enough so that any finite abelian group  $G$  with more than  $m(\varepsilon, k)$  elements must contain either a cyclic subgroup  $H$  of order at least  $n$ , or a subgroup  $H$  which is the direct product of at least  $n'(\varepsilon, k, p)$  cyclic groups of order  $p$  for some prime  $p < n$ .

Now let  $G$  be a finite abelian group with more than  $m(\varepsilon, k)$  elements and let  $A$  be a subset of  $G$  with  $|A| > \varepsilon |G|$ . Let  $H$  be the subgroup whose existence is guaranteed by the choice of  $m(\varepsilon, k)$ . Then  $|A \cap a + H| > \varepsilon |H|$  for some coset  $a + H$  of  $H$ . If  $H$  is cyclic, then  $A - a$  contains the set

$$\{a_0 + c_1 d + c_2(3d) + \dots + c_k(3^{k-1}d) : c_i = 0, 1, 2\},$$

where  $d$  is the difference of the arithmetic progression whose existence is guaranteed by the choice of  $n$  above. If  $H$  is the direct product of at least  $n'(\varepsilon, k, q)$  cyclic groups of order  $p$ , then  $A - a$  contains

$$\{a_0 + c_1 x_1 + \dots + c_k x_k : c_i = 0, 1, 2\}$$

for an independent set of  $x_i$ . Thus in either case  $A$  contains a  $(3, k)$ -lattice and we are finished.

*Remarks.* (1) Roth's special case of Szemerédi's theorem asserts that if  $n$  is sufficiently large and  $A$  is a subset of  $\{1, 2, \dots, n\}$  with more than  $\varepsilon n$  elements then  $A$  contains a set of the form  $\{a, a + x, a + 2x\}$ . This is equivalent to the case  $k = 1$  of the corollary in the case in which  $G$  is cyclic. Indeed, it is not hard to check that one has

$$m(\varepsilon, 1) \leq n \leq \frac{1}{2} m\left(\frac{\varepsilon}{2}, 1\right) + 1$$

(to verify the second inequality consider subsets of the "first half" of a sufficiently large cyclic group). Thus the corollary could be thought of as a group-theoretic generalization of Roth's Theorem.

(2) Since sufficiently large groups contain large abelian subgroups [4], we could actually delete the requirement that  $G$  be abelian in the statement of the corollary.

(3) If the Affine Line Conjecture is valid, then the results here imply the obvious "group-theoretic generalization" of Szemerédi's Theorem: For every  $\varepsilon > 0$ ,  $k$ , and  $l$  there exists an integer  $m(\varepsilon, k, l)$  such that if  $G$  is any finite abelian group with more than  $m(\varepsilon, k, l)$  elements and  $A$  is any subset of  $G$  with more than  $\varepsilon |G|$  elements, then there exists an  $(l, k)$ -lattice in  $A$ .

Finally, we remove the algebraic structure on the underlying set, replacing

$F_q$  with an arbitrary finite set. Thus we consider combinatorial subspaces; we briefly recall the definition (see [6] for further details).

Let  $F$  be the finite set  $\{0, 1, \dots, t-1\}$  with  $t$  elements. A subset  $W$  of  $F^n$  is a *combinatorial  $k$ -space* if it satisfies the following. There is a partition

$$\{1, \dots, n\} = B_0 \cup B_1 \cdots \cup B_k$$

such that  $B_1, \dots, B_k$  are nonempty. There is a function  $f: B_0 \rightarrow F$ . A function  $\bar{f}: F^k \rightarrow F^n$  is defined by  $\bar{f}(y_1, \dots, y_k) = (x_1, \dots, x_n)$  where

$$\begin{aligned} x_i &= f(i) & \text{for } i \text{ in } B_0, \\ x_i &= y_j & \text{for } i \text{ in } B_j, \quad 1 \leq j \leq k. \end{aligned}$$

$W$  is the range of  $\bar{f}$ .

The definition is complicated, but it captures a notion of subspace when the only structure on  $F$  is that of a finite set. We remark that the Hales–Jewett Theorem [6, 7] asserts that if  $n$  is large enough, then in any coloring of  $F^n$  there is a monochromatic combinatorial 1-space (usually called a combinatorial line).

Let  $n''(\varepsilon, k, t)$  be the smallest integer (if one exists) such that if

$$n \geq n''(\varepsilon, k, t), \quad A \subset F^n, \quad |A| > \varepsilon |F^n|,$$

then  $A$  contains a combinatorial  $k$ -space.

**THEOREM 3.** *Let  $t$  be fixed. If  $n''(\varepsilon, 1, t)$  exists for all  $\varepsilon > 0$ , then  $n''(\varepsilon, k, t)$  exists for all  $\varepsilon > 0$  and all  $k$ .*

**COROLLARY.**  *$n''(\varepsilon, k, 2)$  exists for all  $\varepsilon > 0$  and all  $k$ .*

*Proof of the corollary.* The existence of  $n''(\varepsilon, 1, 2)$  is a simple consequence of Sperner's Lemma (see [1] or [6]).

*Remarks.* (1) In [1] it is shown that if there is a fixed  $\varepsilon_0 < 1$  such that  $n''(\varepsilon_0, 1, t)$  exists for all  $t$ , then  $n''(\varepsilon, 1, t)$  exists for all  $\varepsilon > 0$  and all  $t$ . The corresponding result for  $n(\varepsilon, 1, q)$  is proved in [2].

(2) The existence of  $n''(\varepsilon, 1, t)$  is a “density version” of the Hales–Jewett Theorem. Graham has offered a reward for a proof of the existence (or non-existence!) of the numbers  $n''(\varepsilon, 1, 3)$ .

## 2. PROOFS

The following lemma contains the crucial idea underlying Theorems 1, 2, and 3.

**LEMMA.** *Let  $F_q$  be a fixed finite field and  $k$  a fixed positive integer. Assume that  $n(\varepsilon, 1, q)$  exists for all  $\varepsilon > 0$ . Then for each positive integer  $r$ , if  $n(1/(r+1), k, q)$  exists then  $n(1/r, k+1, q)$  exists. Similar statements hold for  $n'(\varepsilon, k, q)$  and  $n''(\varepsilon, k, t)$ .*

*Proof.* We give the proof in the vector space case  $n(\varepsilon, k, q)$ . The proofs for  $n'(\varepsilon, k, q)$  and  $n''(\varepsilon, k, t)$  are entirely analogous. In the lattice case  $n'(\varepsilon, k, q)$  it is merely necessary to replace “ $k$ -space” with “ $(3, k)$ -lattice” and “line” with “ $(3, 1)$ -lattice” throughout. In the combinatorial case  $n''(\varepsilon, k, t)$  it is necessary to replace “affine  $k$ -space” with “combinatorial  $k$ -space” and “affine line” with “combinatorial line” throughout.

Let  $n_0 = n(1/(r+1), k, q)$ . Let  $e$  be the number of distinct  $k$ -dimensional vector subspaces of an  $n_0$ -dimensional vector space over  $F_q$ . Let  $\delta = (q^{n_0} e r^2)^{-1}$  and let  $s = n(\delta, 1, q)$ . We claim that

$$n(1/r, k+1, q) \leq n_0 + s.$$

To prove this we must start with a vector space  $V$  over  $F_q$  of dimension at least  $n_0 + s$ . Let  $A$  be a subset of  $V$  with

$$|A| > (1/r) |V| \geq (1/r) q^{n_0+s}.$$

Let  $W_0$  be a  $n_0$ -dimensional subspace of  $V$  and let

$$V = \bigcup W_\alpha$$

be the decomposition of  $V$  into a union of the pairwise disjoint translates (cosets) of  $W_0$ . For the proof to work in the combinatorial case it is necessary at this point to choose  $W_0$  to be the subspace consisting of the vectors whose last  $s$  components are 0.

Let  $t$  be the number of cosets  $W_\alpha$  such that

$$|A \cap W_\alpha| \leq \frac{1}{r+1} |W_\alpha| = \frac{1}{r+1} q^{n_0}.$$

There are  $q^s$  cosets altogether, so

$$\frac{1}{r} |V| < |A| = \sum |A \cap W_\alpha| \leq \frac{t}{r+1} |W_\alpha| + (q^s - t) |W_\alpha|.$$

This gives

$$q^s - t > q^s/r^2.$$

Hence there are  $d = q^s - t > q^s/r^2$  cosets  $W_\alpha$  such that

$$|A \cap W_\alpha| > \frac{1}{r+1} |W_\alpha|,$$

and since the dimension of  $W_0$  is  $n_0 = n(1/(r+1), k, q)$  each such  $A \cap W_\alpha$  must contain an affine  $k$ -space

$$a_\alpha + U_\alpha,$$

where  $U_\alpha$  is a  $k$ -dimensional vector subspace of  $W_0$ .

Since there are exactly  $e$  distinct  $k$ -dimensional vector subspaces of  $W_0$  at least  $d/e$  of the  $k$ -spaces  $a_\alpha + U_\alpha$  must have the form  $a_\alpha + U$  for a fixed  $U$ . Let these be

$$a_1 + U, \dots, a_h + U,$$

where  $h \geq d/e$ .

Let  $A' = \{a_1, \dots, a_h\}$ . Then

$$|A'| = h \geq d/e > \frac{q^s}{er^2} = \frac{1}{q^{n_0}er^2} q^{n_0+s} = \delta |V|.$$

Since the dimension of  $V$  is  $n_0 + s > s = n(\delta, 1, q)$ , there must be an affine line in  $A'$ . By renumbering if necessary we can assume that this line is  $\{a_1, \dots, a_q\}$ .

It is now easy to check that

$$U' = (a_1 + U) \cup \dots \cup (a_q + U)$$

is an affine  $k+1$ -space contained in  $A$ . Since  $A$  was an arbitrary subset of  $V$  with  $|A| > (1/r)|V|$  this shows that

$$n(1/r, k+1, q) \leq n_0 + s = \dim_{\mathbb{F}_q}(V)$$

as claimed. This finishes the proof of the lemma.

Theorem 1 now follows immediately from the lemma by induction. Indeed, we are given in the hypotheses of the theorem that  $n(\varepsilon, 1, q)$  exists for all  $\varepsilon > 0$ . If  $n(\varepsilon, k, q)$  exists for all  $\varepsilon$ , then it exists for  $\varepsilon = 1/r$ . By the lemma,  $n(\varepsilon, k+1, q)$  exists for all  $\varepsilon > 0$ . Theorem 1 follows by induction on  $k$ .

The proof of Theorem 3 is identical; we merely replace  $n(\varepsilon, k, q)$  with  $n''(\varepsilon, k, r)$ .

To prove Theorem 2 for odd  $q$  we first observe that  $n'(\varepsilon, 1, q)$  exists for all  $\varepsilon > 0$  as a consequence of the main result in [3]. For this case Theorem 2 follows from the lemma and induction as above.

To prove Theorem 2 for even  $q$  we observe that a  $(3, k)$ -lattice is just a  $(2, k)$ -lattice since  $2 = 0$  in  $\mathbb{F}_q$ . It then follows that  $n'(\varepsilon, 1, q)$  exists since any two elements of an abelian group form a  $(2, 1)$ -lattice. The rest of the proof is as above. (An upper bound for  $n'(\varepsilon, k, q)$  for even  $q$  can also be deduced from Lemma 1 in [3].)

*Note added in proof.* The lemma can be easily improved to show that  $n(1/r, k+1, q) \leq n(1/(r+1), k, q) + n(1/(er^2), 1, q)$ .

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